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Analysis of heat transfer through Bi-convection fins

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Abstract

Heat transfer through fins subject to two different convective media is modeled and analyzed analytically in this work. The terminology "Bi-convection fin" is used to refer to this kind of fins. Five different cases are analyzed: Case A: Bi-convection thickness-wise Bi-metallic fins; Case B: Bi-convection span-wise rectangular fins; Case C: Bi-convection longitudinal-wise fins; Case D: Bi-convection perimeter-wise fins with uniform cross-section; and Case E: Bi-convection perimeter-wise permeable fins. Closed form solutions for the fin temperature and heat transfer rate are obtained. It is found that Heat transfer through Bi-convection fins may be minimized for certain designs such as those belonging to cases A, B, D and E. On the other hand, it may be maximized as for those belonging to case C at specific values of fin indices and convection heat transfer coefficients ratio. Finally, the heat transfer through Bi-convection fins is found to increase as their effective thermal conductivity, cross-sectional area, fin indices and the difference between the base and the effective free stream temperatures increase.

Keywords: Fins; Heat transfer; Bi-convection; Conduction; Convection

1. Introduction

The main objective in improving the performance of thermal systems is to enhance heat transfer between hot/cold surfaces and the flowing fluid. Various methods have been proposed to achieve this task. Bergles [1,2] classified these methods as active or passive methods. Active methods are those requiring external power to sustain the enhancement mechanism. For example, imposing a vibrational motion to either the surface or the fluid [3] results in an increase in heat transfer. Other examples for active mechanical augmentation methods are discussed in Ref. [4]. On the other hand, passive enhancement methods do not require external power. Examples of these methods include fins as extended surfaces which is the main interest of this work.

Fins are widely utilized in many industrial applications such as in heat exchanger industry [5–9]. For example, fins are used in air cooled finned tube heat exchangers like car radiators and evaporators and condensers of air conditioning units. Fins are also utilized in cooling of electronic components and gas turbine blades [10]. In these applications, fins can have simple designs such as rectangular, triangular, parabolic, annular, and pin rod fins or complicated designs such as spiral fins [11,12]. In addition, fins can also be arranged in a complex network forming a fin system [13–15]. This kind of fin assemblies is utilized in applications requiring removal of large volumetric heat generation rate [13] such as cooling of large heat flux electronic devices.

To the best knowledge of the author, very few researches in literature considered the variation of the convection coefficient along the fin surface. One of these studies is the work of Laor and Kalman [16] who considered the heat transfer coefficient as power function of temperature. Also, the work of Mokheimer [17] who considered the local variation of the heat transfer coefficient on the upper and lower surfaces of horizontal annular fins with different profiles. On the other hand, there are many situations where fins may have two different convection coefficients as well as two different free stream temperatures. These types of fins are named in this work as "Bi-convection fins". Bi-convection fins can be found in heat exchangers involving condensation or evaporation, cooling or heating of storage tanks and heat transfer through fins separating two immiscible fluids.

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Nomenclature

A_C	cross-sectional area	m^2
Bi	Biot number for hairy fin system	
c_P	specific heat of the fluid	$J\,kg^{-1}\,K^{-1}$
H	Height	m
h_1	convection coefficient for the first convective medium	$W\,m^{-2}\,K^{-1}$
h_{1eff}	effective convection coefficient for the first convective medium	$W\,m^{-2}\,K^{-1}$
h_2	convection coefficient for the second convective medium	$W\,m^{-2}\,K^{-1}$
k	thermal conductivity of the fin	$W\,m^{-1}\,K^{-1}$
L	length	m
m	fin index	m^{-1}
P	pin perimeter	m
q_f	fin heat transfer rate	W
T	fin temperature	K
T_B	base temperature	K
T_C	interface temperature	K
$T_{\infty 1}$	free stream temperature of the first convective medium	K
$T_{\infty 2}$	free stream temperature of the second convective medium	K
$T_{\infty\ eff}$	Effective free stream temperature	K
t	thickness of the primary fin	m
V	velocity across the fin	$m\,s^{-1}$
w	width of the fin	m
x	coordinate axis along the primary fin center line	m
y	thickness-wise coordinate axis	m

Greek symbols

ρ	fluid density	$kg\,m^{-3}$
θ	dimensionless temperature	
Θ	dimensionless fin heat transfer rate	

Subscripts

1	first convective medium
2	second convective medium

In this work, Bi-convection fins are modeled mathematically and heat transfer through them is analyzed analytically. Various designs of Bi-convection fins are analyzed in this work. These are

Case A: Bi-convection thickness-wise Bi-metallic fins,
Case B: Bi-convection span-wise rectangular fins,
Case C: Bi-convection longitudinal-wise fins,
Case D: Bi-convection perimeter-wise fins with uniform cross-section, and
Case E: Bi-convection perimeter-wise permeable fins.

For each case, the governing energy equations are solved analytically and the temperature distribution is determined and appropriate performance indicators are established. As such, extensive parametric study is performed in order to investigate the thermal performance of this kind of fin systems.

2. Problem formulation

2.1. Case A: Bi-convection thickness-wise Bi-metallic fins

Consider a rectangular thin fin extended from a heated surface having a length L and a width w. This fin separates two different convective media. The free stream temperatures for the fluids facing the lower and the upper surfaces are $T_{\infty 1}$ and $T_{\infty 2}$, respectively, as shown in Fig. 1. The convection heat transfer coefficients corresponding to the fluid flows facing the lower and upper surfaces are h_1 and h_2, respectively.

Taking ΔT_1 as the temperature drop across the first layer, the temperature field in the first layer can then approximated as the following by utilizing Taylor expansion:

$$T_1(x, y_1) \cong T_C(x) + \Delta T_1\left(1 - \frac{y_1}{t_1}\right) \tag{1}$$

where x-axis is the axis along the fin length starting from the base surface. The y_1-axis is along the fin thickness-wise direction starting from the surface of the first layer facing the external fluid as illustrated in Fig. 1. T_C is the temperature at the interface between the layers. t_1 is the thickness of the first layer. Since the heat flux at the interface is the same for both layers, $k_2(\Delta T_2/t_2) \cong k_1(\Delta T_1/t_1)$. The temperature drop in the second layer ΔT_2 is approximated by $\Delta T_2 \cong \Delta T_1[(t_2/t_1)(k_1/k_2)]$. As such, the temperature field in the second layer is

$$T_2(x, y_2) \cong T_C(x) - \Delta T_1\left[\left(\frac{t_2}{t_1}\right)\left(\frac{k_1}{k_2}\right)\right]\left(\frac{y_2}{t_2}\right) \tag{2}$$

where the y_2-axis is along the fin thickness-wise direction starting from the interface, t_2 is the thickness of the second layer.

The quantity ΔT_1 is assumed to be constant so that one-dimensional analysis can be utilized to estimate $T_C(x)$. ΔT_1 is obtained by the fact that the heat transfer across the fin far from the base is equal to $T_{\infty 1} - T_{\infty 2}$ divided by the total thermal resistance. As such, ΔT_1 is equal to

$$\Delta T_1 = \frac{T_{\infty 1} - T_{\infty 2}}{\frac{1}{h_1} + \frac{t_1}{k_1} + \frac{t_2}{k_2} + \frac{1}{h_2}}\left(\frac{t_1}{k_1}\right) = \frac{T_{\infty 1} - T_{\infty 2}}{\frac{1}{Bi_1}\left[\left(\frac{h_1}{h_2}\right) + 1\right] + \left(\frac{k_1}{k_2}\right)\left(\frac{t_2}{t_1}\right) + 1} \tag{3}$$

where Bi_1 is the Biot number of the first layer ($Bi_1 = h_1 t_1/k_1$).

The application of the conservation of energy principle to fin differential elements in the first and second layers of thickness Δx results in the following equations:

$$q_{1x} + h_1 w\big(T_{\infty 1} - T_1(x, y_1 = 0)\big)\Delta x = q_{1x} + \frac{dq_{1x}}{dx}\Delta x + q_i'' w\Delta x \tag{4}$$

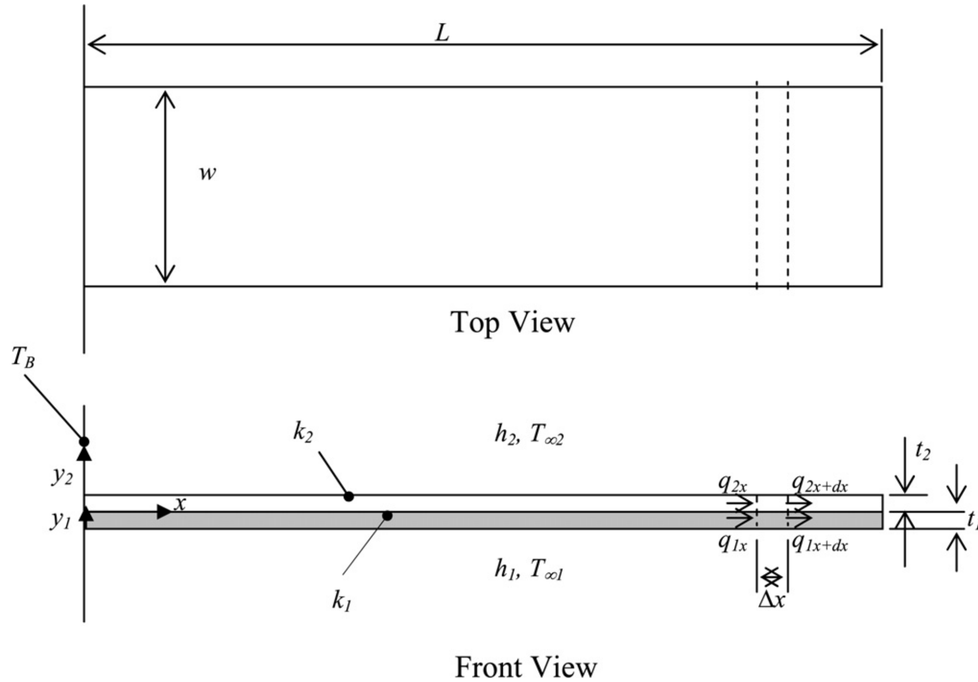

Fig. 1. Schematic diagram for the Bi-convection depth-wise Bi-metallic fin (case A).

$$q_{2x} + q_i'' w \Delta x = q_{2x} + \frac{dq_{2x}}{dx}\Delta x + h_2 w\big(T_2(x, y_2 = t_2) - T_{\infty 2}\big)\Delta x \tag{5}$$

where q_i'' is the heat flux at the interface. The terms q_{1x} and q_{2x} are the conduction heat transfer at the left surfaces of the differential elements. They are equal to $q_{1x} = k_1 t_1 w\, dT_C/dx$ and $q_{2x} = k_2 t_2 w\, dT_C/dx$, respectively.

The summation of Eqs. (4) and (5) can be arranged in the following differential equation form:

$$\frac{d^2 T_C}{dx^2} - m^2(T_C - T_{\infty,\mathrm{eff}}) = 0 \tag{6}$$

where m and $T_{\infty,\mathrm{eff}}$ are the fin index and a quantity representing an effective value for a free stream temperatures. They are equal to

$$m = \sqrt{\frac{h_1 + h_2}{k_1 t_1 + k_2 t_2}} \tag{7a}$$

$$T_{\infty,\mathrm{eff}} = \frac{T_{\infty 1} + (h_2/h_1)T_{\infty 2}}{1 + (h_2/h_1)} + \Delta T_1\left[\frac{Bi_2/Bi_1 - 1}{1 + h_2/h_1}\right] \tag{7b}$$

where $Bi_2 = h_2 t_2 / k_2$.

The dimensionless effective free stream temperature $\theta_{\infty\ \mathrm{eff}}$ is defined as

$$\theta_{\infty\ \mathrm{eff}} = \frac{T_{\infty\ \mathrm{eff}} - T_{\infty 2}}{T_{\infty 1} - T_{\infty 2}} = \frac{1}{1 + h_2/h_1} + \frac{(Bi_2 - Bi_1)/(1 + h_2/h_1)}{\big[\big(\frac{h_1}{h_2}\big) + 1\big] + \big(\frac{k_1}{k_2}\big)\big(\frac{t_2}{t_1}\big)Bi_1 + Bi_1} \tag{8}$$

For situations when $Bi_1 \ll 1$. Eq. (8) can be reduced to the following

$$\theta_{\infty\ \mathrm{eff}} \cong \frac{1}{1 + h_2/h_1} + \frac{(Bi_2 - Bi_1)}{(h_2/h_1)(h_1/h_2 + 1)^2} \tag{9}$$

For a very large fin, dT_C/dx approaches zero at the tip of the fin. As such, the solution of Eq. (6) is

$$\frac{T_C(x) - T_{\infty,\mathrm{eff}}}{T_B - T_{\infty,\mathrm{eff}}} = e^{-mx} \tag{10}$$

where T_B is the base temperature. For this case, the fin dissipates the maximum heat rate and it is equal to

$$q_f = -(k_1 t_1 + k_2 t_2) w \frac{dT_C}{dx}\bigg|_{x=0} = w\sqrt{(k_1 t_1 + k_2 t_2)(h_1 + h_2)}\{T_B - T_{\infty,\mathrm{eff}}\} \tag{11}$$

The dimensionless heat transfer rate, Θ, is related to the dimensionless effective free stream temperature through the following relation:

$$\Theta = \frac{q_f}{w\sqrt{(k_1 t_1 + k_2 t_2)h_1}\{T_B - T_{\infty 1}\}} = \left(1 - \left[\frac{T_{\infty 2} - T_{\infty 1}}{T_B - T_{\infty 1}}\right](1 - \theta_{\infty\ \mathrm{eff}})\right)\left[\frac{Bi_2 - Bi_1 + 1}{2\theta_{\infty\ \mathrm{eff}}}\right]^{1/2} \times\left[1 + \sqrt{1 - \frac{4(Bi_2 - Bi_1)\theta_{\infty\ \mathrm{eff}}}{(Bi_2 - Bi_1 + 1)^2}}\right]^{1/2} \tag{12}$$

2.2. Case B: Bi-convection span-wise rectangular fins

Consider a rectangular thin fin having a thermal conductivity k, length L and a thickness t. The fin surface is subject to two different convection conditions as shown in Fig. 2. The first region has a span height H_1 while the second region has a span height H_2. The convection medium facing the first region has $T_{\infty 1}$ and h_1 as the free stream temperature and the heat transfer convection coefficient, respectively, while $T_{\infty 2}$ and h_2 are those corresponding to the convection conditions for the second region.

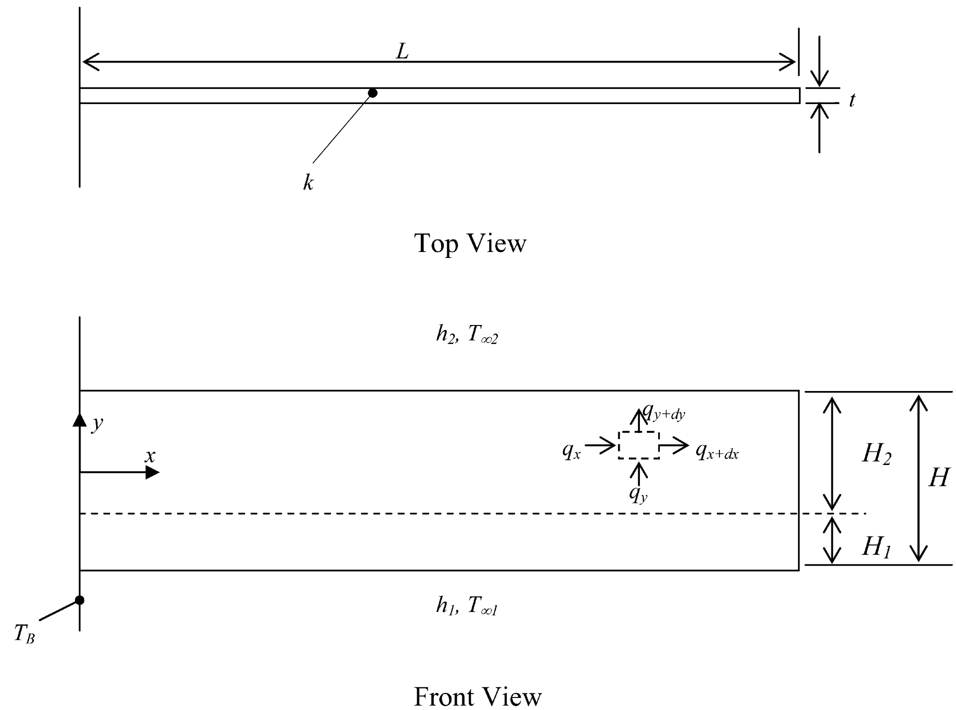

Fig. 2. Schematic diagram for the Bi-convection span-wise rectangular fin (case B).

Assuming the temperature does not vary along thickness the fin, the application of the conservation of energy prinicple results in the following equations:

$$q_x + q_y = q_x + q_y + \frac{dq_x}{dx}\Delta x + \frac{dq_y}{dy}\Delta y + 2h_1\big(T_1(x,y) - T_{\infty 1}\big)\Delta x\Delta y, \quad y < 0 \tag{13}$$

$$q_x + q_y = q_x + q_y + \frac{dq_x}{dx}\Delta x + \frac{dq_y}{dy}\Delta y + 2h_2\big(T_2(x,y) - T_{\infty 2}\big)\Delta x\Delta y, \quad y \geqslant 0 \tag{14}$$

where the x-axis is along the fin length and y is along the fin span height starting from the interface between the two convective media. T_1 and T_2 are the temperature field of the first and the second portions of the fin, respectively. The terms q_x and q_y are the conduction heat transfer at the left surface and the bottom surface of the differential element shown in Fig. 2. They are equal to $q_x = k\Delta y t\, dT_{1,2}/dx$ and $q_y = k\Delta x t\, dT_{1,2}/dy$, respectively.

Eqs. (13) and (14) can be reduced to the following partial differential equations:

$$\frac{d^2T_1}{dx^2} + \frac{d^2T_1}{dy^2} - \frac{2h_1}{kt}(T_1 - T_{\infty 1}) = 0, \quad y < 0 \tag{15}$$

$$\frac{d^2T_2}{dx^2} + \frac{d^2T_2}{dy^2} - \frac{2h_2}{kt}(T_2 - T_{\infty 2}) = 0, \quad y \geqslant 0 \tag{16}$$

The boundary conditions are

$$T_1(x=0,y) = T_2(x=0,y) = T_B \tag{17}$$

$$\left.\frac{\partial T_1}{\partial x}\right|_{x\to\infty} = \left.\frac{\partial T_2}{\partial x}\right|_{x\to\infty} = 0 \tag{18}$$

$$\left.\frac{\partial T_1}{\partial y}\right|_{y=0} = \left.\frac{\partial T_2}{\partial y}\right|_{y=0}; \qquad T_1(x,y=0) = T_2(x,y=0) \tag{19}$$

$$\left.\frac{\partial T_1}{\partial y}\right|_{y=-H_1} = \left.\frac{\partial T_2}{\partial y}\right|_{y=H_2} = 0 \tag{20}$$

Utilizing separation of variables, the temperature field can be found to be equal to the following:

$$\begin{aligned} T_1(x,y) &= T_{\infty 1} + (T_B - T_{\infty 1})\left\{\frac{\cosh[m_1(L-x)]}{\cosh(m_1 L)}\right\} \\ &+ \left\{\frac{(T_{\infty 1} - T_{\infty 2})\cosh[m_1(y+H_1)]}{\cosh(m_1H_1) + (m_1/m_2)[\sinh(m_1H_1)/\tanh(m_2H_2)]}\right\} \\ &\times \left\{\frac{\cosh[m_1(L-x)]}{\cosh(m_1 L)}\right\} \\ &- \left\{\frac{(T_{\infty 1} - T_{\infty 2})\cosh[m_1(y+H_1)]}{\cosh(m_1H_1) + (m_1/m_2)[\sinh(m_1H_1)/\tanh(m_2H_2)]}\right\} \end{aligned} \tag{21}$$

$$\begin{aligned} T_2(x,y) &= T_{\infty 2} + (T_B - T_{\infty 2})\left\{\frac{\cosh[m_2(L-x)]}{\cosh(m_2 L)}\right\} \\ &+ \left\{\frac{(T_{\infty 1} - T_{\infty 2})\cosh[m_2(y-H_2)]}{\cosh(m_2H_2) + (m_2/m_1)[\sinh(m_2H_2)/\tanh(m_1H_1)]}\right\} \\ &\times \left\{\frac{\cosh[m_2(L-x)]}{\cosh(m_2 L)}\right\} \\ &- \left\{\frac{(T_{\infty 1} - T_{\infty 2})\cosh[m_2(y-H_2)]}{\cosh(m_2H_2) + (m_2/m_1)[\sinh(m_2H_2)/\tanh(m_1H_1)]}\right\} \end{aligned} \tag{22}$$

where $m_1 = \sqrt{2h_1/(kt)}$ and $m_2 = \sqrt{2h_2/(kt)}$.

The dimensionless fin heat transfer rate Θ is

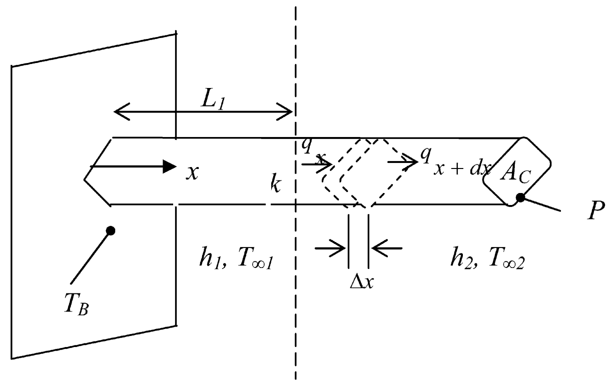

Fig. 3. Schematic diagram for the Bi-convection longitudinal-wise fin (case C).

$$\Theta = \frac{q_f}{kt(T_{\infty 1} - T_{\infty 2})} = \frac{-\int_{-H_1}^{0} \frac{\partial T_1}{\partial x}\big|_{x=0} dy - \int_{0}^{H_2} \frac{\partial T_2}{\partial x}\big|_{x=0} dy}{(T_{\infty 1} - T_{\infty 2})}$$
$$= m_2 H_2 - (m_2 H_2 + m_1 H_1)\theta_{1B} + \frac{\left(\frac{h_2}{h_1} - 1\right)\coth(m_1 H_1) - \left(\frac{h_1}{h_2} - 1\right)\coth(m_2 H_2)}{\left(\sqrt{\frac{h_2}{h_1}}\coth(m_1 H_1) + \sqrt{\frac{h_1}{h_2}}\coth(m_2 H_2)\right)^2} \tag{23}$$

where

$$\theta_{1B} = \frac{T_B - T_{\infty 1}}{T_{\infty 2} - T_{\infty 1}} \tag{24}$$

When $m_1 H_1 > 2.65$ and $m_2 H_2 > 2.65$, Eq. (23) reduces to the following:

$$\Theta = m_2 H_2 - (m_1 H_1 + m_2 H_2)\theta_{1B} + \left\{\frac{\sqrt{h_2/h_1} - 1}{\sqrt{h_2/h_1} + 1}\right\} \tag{25}$$

The maximum and minimum dimensionless fin heat transfer rates are

$$\Theta_{\max} = m_2 H_2 - (m_1 H_1 + m_2 H_2)\theta_{1B} + 1 \quad \text{when } \frac{h_2}{h_1} \to \infty \tag{26}$$

$$\Theta_{\min} = m_2 H_2 - (m_1 H_1 + m_2 H_2)\theta_{1B} - 1 \quad \text{when } \frac{h_2}{h_1} = 0 \tag{27}$$

2.3. Case C: Bi-convection longitudinal-wise fins

Consider a fin with uniform cross-section A_C and perimeter P having a thermal conductivity k and a very long length. The fin portion starting from the base and ending at a distance L_1 from the base is subject to convection with a free stream temperature of $T_{\infty 1}$ and a convection heat transfer coefficient h_1. The remaining portion is subject to another convective medium with $T_{\infty 2}$ and h_2 as the convective parameters as shown in Fig. 3.

Assuming the temperature does not vary along thickness the fin, the application of the conservation of energy prinicipe results in the following equations:

$$q_x = q_x + \frac{dq_x}{dx}\Delta x + h_1 P\big(T_1(x,y) - T_{\infty 1}\big)\Delta x, \quad x < L_1 \tag{28}$$

$$q_x = q_x + \frac{dq_x}{dx}\Delta x + h_2 P_2\big(T_2(x,y) - T_{\infty 2}\big)\Delta x, \quad x \geqslant L_2 \tag{29}$$

where the x-axis is along the fin length. T_1 and T_2 are the temperature field of the first and the second portions of the fin, respectively. The terms q_x is the conduction heat transfer at the left surface of the differential element shown in Fig. 3. It is equal to $q_x = kA_C\, dT_{1,2}/dx$.

Eqs. (28) and (29) can be reduced to the following ordinary differential equations:

$$\frac{d^2 T_1}{dx^2} - \frac{h_1 P}{kA_C}(T_1 - T_{\infty 1}) = 0, \quad x < L_1 \tag{30}$$

$$\frac{d^2 T_2}{dx^2} - \frac{h_2 P}{kA_C}(T_2 - T_{\infty 2}) = 0, \quad x \geqslant L_1 \tag{31}$$

where x-axis is along the fin length starting from the base. T_1 and T_2 are the temperature field of the first and the second portions of the fin, respectively. The boundary conditions are

$$T_1(x = 0) = T_B; \qquad T_2(x \to \infty) = T_{\infty 2} \tag{32}$$

$$\frac{dT_1}{dx}\bigg|_{x=L_1} = \frac{dT_2}{dx}\bigg|_{x=L_1} \tag{33}$$

$$T_1(x = L_1) = T_2(x = L_1) \tag{34}$$

The solution of Eqs. (30) and (31) subject to the listed boundary conditions can be arranged in the following forms:

$$\theta_1(x) = \frac{T_1(x) - T_{\infty_1}}{T_{\infty 2} - T_{\infty 1}} = \left(\frac{T_B - T_{\infty_1}}{T_{\infty 2} - T_{\infty 1}}\right)\cosh(m_1 x) + \left\{\frac{\operatorname{sech}(m_1 L_1) - \left[\frac{(T_B - T_{\infty 1})}{(T_{\infty_2} - T_{\infty 1})}\right]\left[1 + \sqrt{h_1/h_2}\tanh(m_1 L_1)\right]}{\tanh(m_1 L_1) + \sqrt{h_1/h_2}}\right\} \times \sinh(m_1 x) \tag{35}$$

$$\theta_2(x) = \frac{T_2(x) - T_{\infty 2}}{T_{\infty 1} - T_{\infty 2}} = \left\{\frac{1 - \left[\frac{(T_B - T_{\infty 1})}{T_{\infty 2} - T_{\infty 1}}\right]\operatorname{sech}(m_1 L_1)}{\tanh(m_1 L_1) + \sqrt{h_1/h_2}}\right\}\sqrt{\frac{h_1}{h_2}}\, e^{-m_2(x - L_1)} \tag{36}$$

where $m_1 = \sqrt{h_1 P/(kA_C)}$ and $m_2 = \sqrt{h_2 P/(kA_C)}$. The dimensionless fin heat transfer rate at the base Θ is

$$\Theta = \frac{-kA_C dT_1/dx|_{x=0}}{\sqrt{kA_C P h_1}(T_B - T_{\infty 1})} = \frac{1 + \sqrt{\frac{h_1}{h_2}}\tanh(m_1 L_1) - \left(\frac{T_{\infty 1} - T_{\infty 2}}{T_{\infty 1} - T_B}\right)\operatorname{sech}(m_1 L_1)}{\tanh(m_1 L_1) + \sqrt{\frac{h_1}{h_2}}} \tag{37}$$

2.4. Case D: Bi-convection perimeter-wise fins with uniform cross-section

Consider a fin with uniform cross-section A_C having a thermal conductivity k and a very long length. A uniform fin portion of perimeter P_1 is subject to convection with a free stream temperature of $T_{\infty 1}$ and a convection heat transfer coefficient of h_1 while the remaining fin portion of perimeter P_2 is subject to another convection medium with $T_{\infty 2}$ and h_2 as the convective parameters as shown in Fig. 4. The free stream temperatures

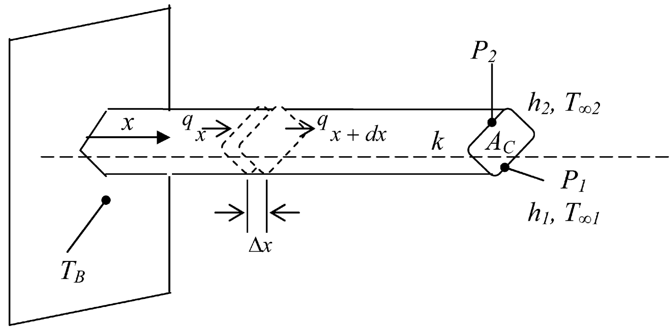

Fig. 4. Schematic diagram for the Bi-convection perimeter-wise fin with uniform cross-section (case D).

$T_{\infty 1}$ and $T_{\infty 2}$ are considered to vary linearly with x as when the free stream fluid is subject to conduction with the main surface. They can be described according to the following model:

$$T_{\infty 1} = T_{\infty 1o} + ax \tag{38}$$

$$T_{\infty 2} = T_{\infty 2o} + bx \tag{39}$$

The application of the conservation of energy principle to a differential element of thickness Δx results in the following:

$$q_x = q_x + \frac{dq_x}{dx}\Delta x + h_1 P_1 (T - T_{\infty 1})\Delta x + h_2 P_2 (T - T_{\infty 2})\Delta x \tag{40}$$

where T is the temperature field and $q_x = -kA_C dT/dx$. Eq. (40) is derived based on the assumption that the temperature variation along the cross-section is negligible. That is, $Bi_1 = h_1 A_C/(kP_1) \ll 1$ and $Bi_2 = h_2 A_C/(kP_2) \ll 1$, Eq. (40) can be reduced to the following differential equation:

$$\frac{d^2T}{dx^2} - \frac{h_1 P_1}{kA_C}(T - T_{\infty 1o}) - \frac{h_2 P_2}{kA_C}(T - T_{\infty 2o}) = -\left(\frac{h_1 P_1 a}{kA_C} + \frac{h_2 P_2 b}{kA_C}\right)x \tag{41}$$

The boundary conditions are

$$T(x=0) = T_B; \qquad \left.\frac{dT}{dx}\right|_{x\to\infty} = 0 \tag{42}$$

The solution of Eq. (41) subject to boundary conditions (42) can be arranged in the following forms:

$$\theta(x) = \frac{T(x) - T_{\infty\ \mathrm{eff}}}{T_B - T_{\infty \mathrm{eff}}} = e^{-mx} + \frac{x}{(T_B - T_{\infty\ \mathrm{eff}})}\left\{\frac{a}{(1 + h_2 P_2/[h_1 P_1])} + \frac{b}{(1 + h_1 P_1/[h_2 P_2])}\right\} \tag{43}$$

where

$$m = \sqrt{\frac{h_1 P_1 + h_2 P_2}{kA_C}} \tag{44}$$

$$T_{\infty\ \mathrm{eff}} = \frac{T_{\infty 1o} + (h_2 P_2/[h_1 P_1])T_{\infty 2o}}{1 + (h_2 P_2/[h_1 P_1])} \tag{45}$$

Define the dimensionless effective free stream temperature as follows

$$\theta_{\infty\ \mathrm{eff}} = \frac{T_{\infty\ \mathrm{eff}} - T_{\infty 2o}}{T_{\infty 1o} - T_{\infty 2o}} = \frac{1}{1 + (h_2 P_2/[h_1 P_1])} \tag{46}$$

The dimensionless fin heat transfer rate at the base Θ is

$$\Theta = \frac{-kA_C dT/dx|_{x=0}}{\sqrt{kA_C h_1 P_1}(T_B - T_{\infty 1o})} = \frac{1}{\sqrt{\theta_{\infty\ \mathrm{eff}}}} - \frac{(1 - \theta_{\infty\ \mathrm{eff}})}{\sqrt{\theta_{\infty\ \mathrm{eff}}}}\left(\frac{T_{\infty 2o} - T_{\infty 1o}}{T_B - T_{\infty 1o}}\right) - \left\{\frac{a\theta_{\infty\ \mathrm{eff}}}{(T_B - T_{\infty 1o})}\right\}\sqrt{\frac{kA_C}{h_1 P_1}} - \left\{\frac{b(1 - \theta_{\infty\ \mathrm{eff}})}{(T_B - T_{\infty 1o})}\right\}\sqrt{\frac{kA_C}{h_1 P_1}} \tag{47}$$

When $h_2 = h_1$ and $T_{\infty 1} = T_{\infty 2}$, Eq. (47) reduces to

$$\Theta = \frac{-kA_C dT/dx|_{x=0}}{\sqrt{kA_C hP}(T_B - T_{\infty 1o})} = 1 - \left\{\frac{a}{(T_B - T_{\infty 1o})}\right\}\sqrt{\frac{kA_C}{hP}} \tag{48}$$

2.5. Case E: Bi-convection perimeter-wise permeable fins

Consider a fin with uniform cross-section A_C and a perimeter P having a thermal conductivity k and a very long length. The fin is permeable and encounter uniform flow across it with a speed of V. The lower surface of the fin is subject to convection with a free stream temperature of $T_{\infty 1}$ and a convection coefficient h_1 while the upper surface is subject to another convection medium with convection parameters $T_{\infty 2}$ and h_2 as shown in Fig. 5. The direction of V is from the lower surface to the upper surface of the fin.

Assuming the temperature variation along the cross-section is negligible, the energy equation, Eq. (41) is modified to the following:

$$\frac{d^2T}{dx^2} - \frac{h_1 P_1}{kA_C}(T - T_{\infty 1}) - \frac{h_2 P_2}{kA_C}(T - T_{\infty 2}) - \frac{\rho c_P V}{kA_C}(T - T_{\infty 1}) = 0 \tag{49}$$

where ρ, c_P, P_1, P_2 and T are the lower fluid density, the lower fluid specific heat, perimeter of the lower fin portion, perimeter of the upper fin portion and the temperature. The boundary conditions are

$$T(x=0) = T_B; \qquad \left.\frac{dT}{dx}\right|_{x\to\infty} = 0 \tag{50}$$

The solution of Eq. (49) subject to boundary conditions (50) can be arranged in the following form:

$$\theta(x) = \frac{T(x) - T_{\infty\ \mathrm{eff}}}{T_B - T_{\infty\ \mathrm{eff}}} = e^{-mx} \tag{51}$$

where

$$m = \sqrt{\frac{h_{1\mathrm{eff}} P_1 + h_2 P_2}{kA_C}} \tag{52}$$

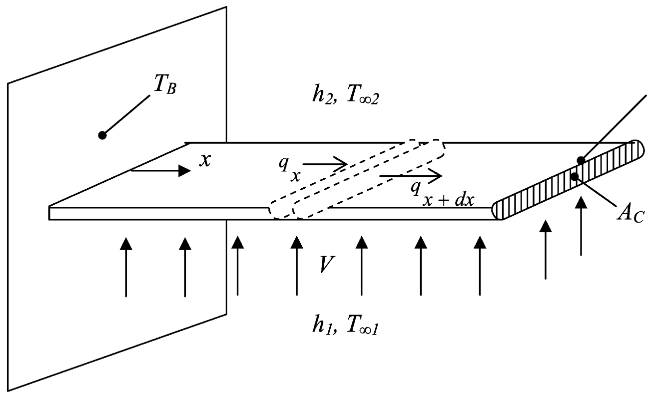

Fig. 5. Schematic diagram for the Bi-convection perimeter-wise permeable fin (case E).

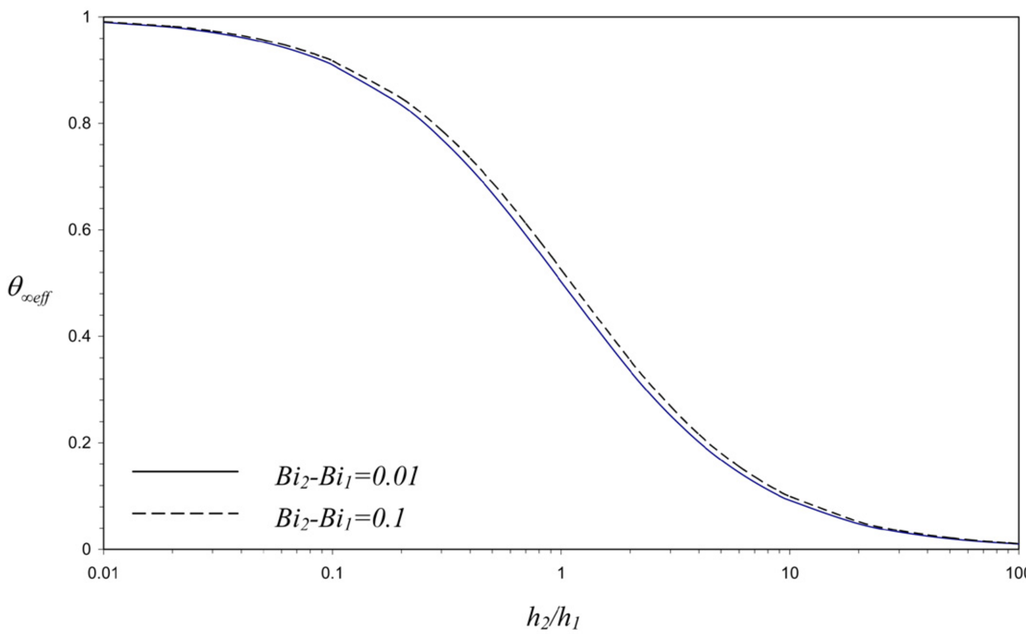

Fig. 6. Effect of convection coefficient ratio h_2/h_1 on the dimensionless effective free stream temperature $\theta_{\infty\ \mathrm{eff}}$ for case A.

$$T_{\infty\ \mathrm{eff}} = \frac{T_{\infty 1} + (h_2 P_2/[h_{1\mathrm{eff}} P_1])T_{\infty 2}}{1 + (h_2 P_2/[h_{1\mathrm{eff}} P_1])} \tag{53}$$

$$h_{1\mathrm{eff}} = h_1 + \rho c_P V \tag{54}$$

Define the dimensionless effective free stream temperature as follows

$$\theta_{\infty\ \mathrm{eff}} = \frac{T_{\infty\ \mathrm{eff}} - T_{\infty 2}}{T_{\infty 1} - T_{\infty 2}} = \frac{1}{1 + (h_2 P_2/[h_{1\mathrm{eff}} P_1])} \tag{55}$$

The dimensionless fin heat transfer rate at the base Θ is

$$\Theta = \frac{-kA_C dT/dx|_{x=0}}{\sqrt{kA_C h_{1\mathrm{eff}} P_1}(T_B - T_{\infty 1})} = \frac{1}{\sqrt{\theta_{\infty\ \mathrm{eff}}}} - \frac{(1-\theta_{\infty\ \mathrm{eff}})}{\sqrt{\theta_{\infty\ \mathrm{eff}}}}\left(\frac{T_{\infty 2} - T_{\infty 1}}{T_B - T_{\infty 1}}\right) \tag{56}$$

The dimensionless fin heat transfer rate is minimized when

$$T_{\infty\ \mathrm{eff}} = T^*_{\infty\ \mathrm{eff}} = 2T_{\infty 2} - T_B \tag{57}$$

provided that

$$T_{\infty 2} < T_B < 2T_{\infty 2} - T_{\infty 1} \quad \text{and} \quad T_{\infty 2} > T_{\infty 1} \tag{58}$$

For this condition the minimum dimensionless heat transfer is equal to

$$(q_f)_{\min} = 2\sqrt{kA_C h_{1\mathrm{eff}} P_1 (T_{\infty 2} - T_{\infty 1})(T_B - T_{\infty 2})} \tag{59}$$

This necessitates that

$$\frac{h_2 P_2}{h_{1\mathrm{eff}} P_1} = \frac{T_{\infty 1} + T_B - 2T_{\infty 2}}{T_{\infty 2} - T_B} \tag{60}$$

3. Discussion of the results

Fig. 6 illustrates the effect of the convection ratio h_2/h_1 and the differential Biot numbers Bi_2–Bi_1 on the dimensionless effective free stream temperature $\theta_{\infty\ \mathrm{eff}}$ for case A. As h_2/h_1 increases, $T_{\infty\ \mathrm{eff}}$ approaches $T_{\infty 2}$ thus, $\theta_{\infty\ \mathrm{eff}}$ decreases and approaches zero. On the other hand, $T_{\infty\ \mathrm{eff}}$ approaches $T_{\infty 1}$ as h_2/h_1 approaches zero and therefore $\theta_{\infty\ \mathrm{eff}}$ approaches unity as shown in Fig. 6. As Bi_2–Bi_1 increases, the conduction thermal resistance increases thus, $T_{\infty\ \mathrm{eff}}$ increases and approaches $T_{\infty 1}$. That is why $\theta_{\infty\ \mathrm{eff}}$ increases as Bi_2–Bi_1 increases as shown in Fig. 6.

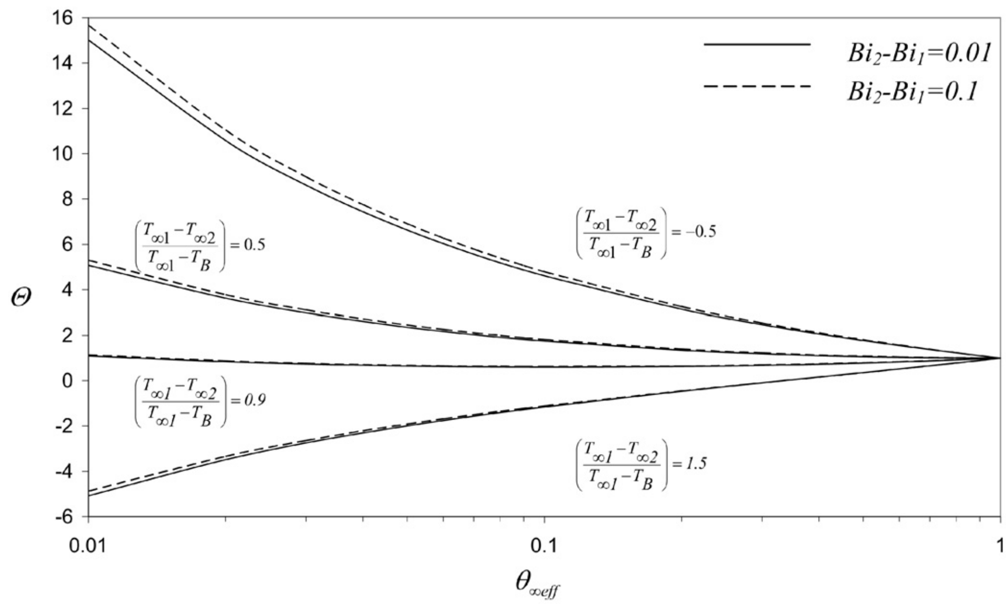

Fig. 7. Effect of on the dimensionless effective free stream temperature $\theta_{\infty\ \mathrm{eff}}$ on the dimensionless heat transfer Θ for case A.

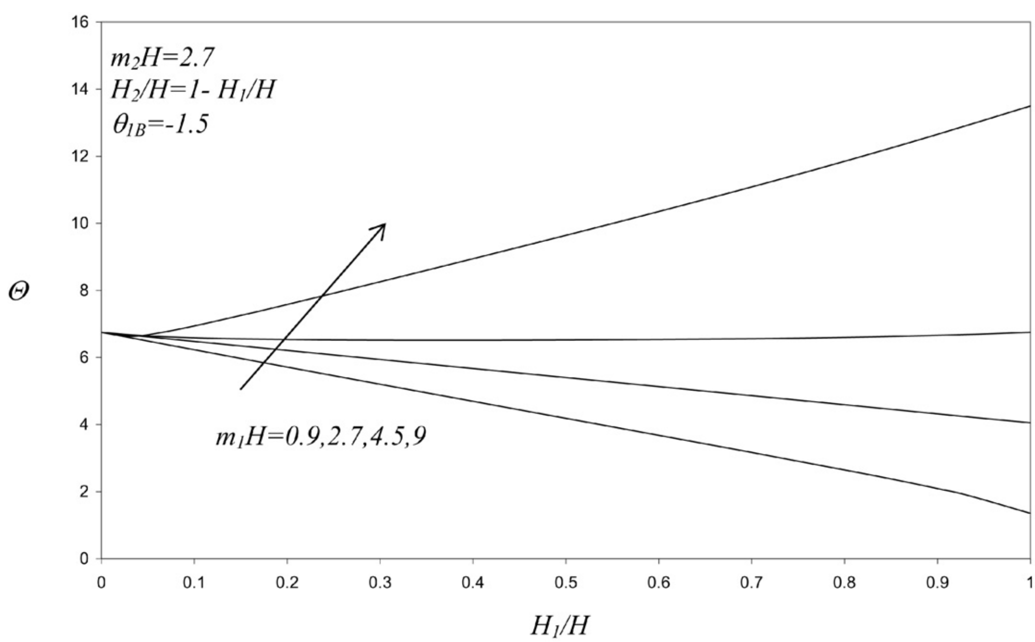

Fig. 8. Effects of the relative width $H_1/(H_1 + H_2)$ and the fin index m_1 on the dimensionless fin heat transfer rate Θ for case B.

Fig. 7 shows that as $\theta_{\infty\ \mathrm{eff}}$ decreases for case A, the dimensionless heat transfer rate Θ increases when $(T_{\infty 1} - T_{\infty 2})/(T_{\infty 1} - T_B)$ is negative. This is satisfied when $T_{\infty 2} < T_{\infty 1} < T_B$ or when $T_B < T_{\infty 1} < T_{\infty 2}$. The former and latter conditions may be satisfied in evaporation and condensation applications, respectively. For these conditions, the decrease in $\theta_{\infty\ \mathrm{eff}}$ results in an increase in Θ as the absolute value of fin excess temperature $|T_B - T_{\infty\ \mathrm{eff}}|$ increases as $\theta_{\infty\ \mathrm{eff}}$ decreases.

When $(T_{\infty 1} - T_{\infty 2})/(T_{\infty 1} - T_B)$ is positive as when $T_{\infty 1} < T_{\infty 2}$ and $T_{\infty 1} < T_B$ or when $T_{\infty 1} > T_{\infty 2}$ and $T_{\infty 1} > T_B$, the dimensionless fin heat transfer rate Θ may have positive or negative values as Fig. 7 shows. Accordingly, Θ can be minimized at a specific value of $\theta_{\infty\ \mathrm{eff}}$. The former and latter conditions may be satisfied for side plates of heated or cooled storage tanks, respectively. It is worth noting that Θ is positive when $T_B > T_{\infty\ \mathrm{eff}}$ while it is negative when $T_B < T_{\infty\ \mathrm{eff}}$.

Fig. 8 shows the effect of the relative height span of the first convective medium H_1/H and the fin index m_1 on the dimensionless fin heat transfer rate Θ for case B. As m_1 increases, Θ increases when $H_1/H > 0.067$. It is noticed that Θ has local minimum a specific value of H_1/H especially when $m_1 H > 2.7$ for the selected parameters. This indicates that special care should be considered in selecting the spans H_1 and H_2 for the case B Bi-convection fins. Typical applications for this kind of Bi-convection fins can be found in condensation and evaporation applications. To avoid minimization in Θ, H_1 and

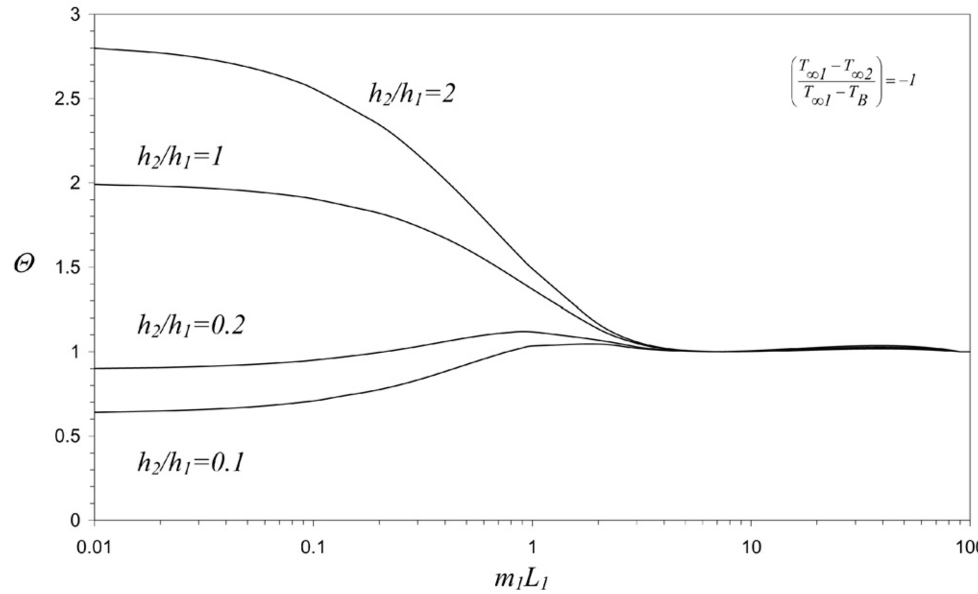

Fig. 9. Effects of the fin index m_1 and convection coefficient ratio h_2/h_1 on the dimensionless fin heat transfer rate Θ for case C.

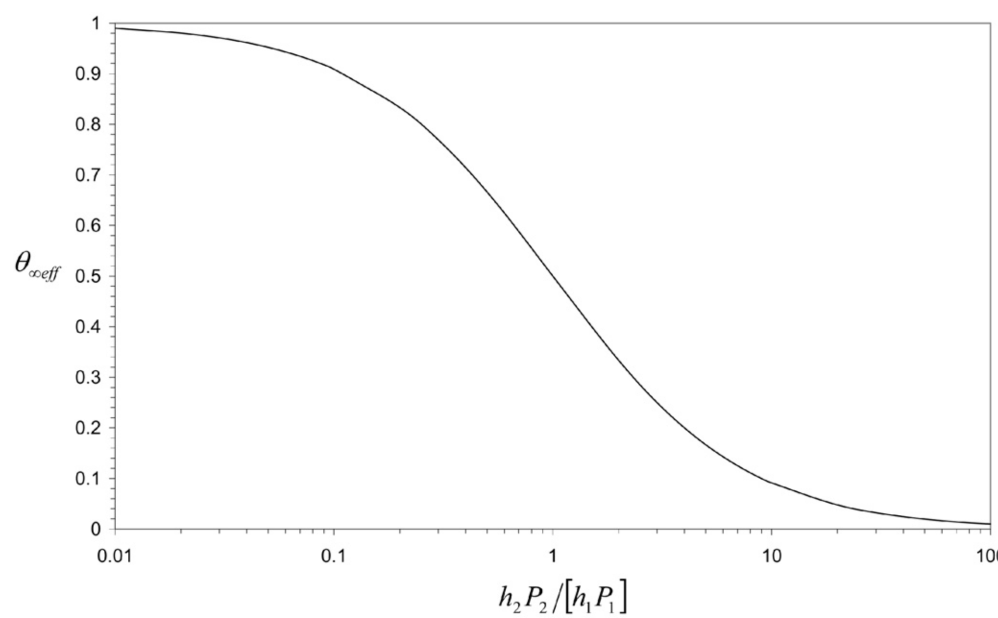

Fig. 10. Effect of convection heat transfer ratio $h_2P_2/[h_1P_1]$ the dimensionless effective free stream temperatures $\theta_{\infty\ \mathrm{eff}}$ for cases D and E.

H_2 are recommended to be greater than $2.65/m_1$ and $2.65/m_2$, respectively. For these conditions, the dimensionless fin heat transfer rate Θ is calculated from Eq. (25).

Fig. 9 describes the effect of case C Bi-convection fin parameter m_1L_1 and the convection ratio h_2/h_1 on the dimensionless fin heat transfer rate Θ. For condensation applications where $((T_{\infty1} - T_{\infty2})/(T_{\infty1} - T_B))$ is negative, Θ is maximized at certain values of h_2/h_1 and m_1L_1 especially when $h_2/h_1 < 1.0$ as illustrated in Fig. 9.

Fig. 10 shows that $\theta_{\infty\ \mathrm{eff}}$ increases until approaches unity as $h_2P_2/[h_1P_1]$ decreases and approaches zero for cases D and E Bi-convection fins. Thus, $T_{\infty\ \mathrm{eff}}$ approaches $T_{\infty1}$. On the hand, $T_{\infty\ \mathrm{eff}}$ approaches $T_{\infty2}$ as $h_2P_2/[h_1P_1]$ increases and approaches infinity. For case E, Θ increases as $\theta_{\infty\ \mathrm{eff}}$ decreases for negative values of $(T_{\infty2} - T_{\infty1})/(T_B - T_{\infty1})$ as shown in Fig. 11. Negative values of $(T_{\infty2} - T_{\infty1})/(T_B - T_{\infty1})$ are found in condensation applications where the temperature of the condensate $T_{\infty1}$ is higher than the temperature of base T_B and lower than the temperature of the vapor $T_{\infty2}$. As $T_{\infty\ \mathrm{eff}}$ approaches $T_{\infty2}$ as when $\theta_{\infty\ \mathrm{eff}}$ decreases, the excess temperature $|T_B - T_{\infty\ \mathrm{eff}}|$ increases. To avoid minimization in Θ when $(T_{\infty2} - T_{\infty1})/(T_B - T_{\infty1})$ is positive, the convection coefficient ratio should satisfy the following inequality:

$$\frac{h_2P_2}{h_{1\mathrm{eff}}P_1} > \frac{T_{\infty1} + T_B - 2T_{\infty2}}{T_{\infty2} - T_B} \tag{61}$$

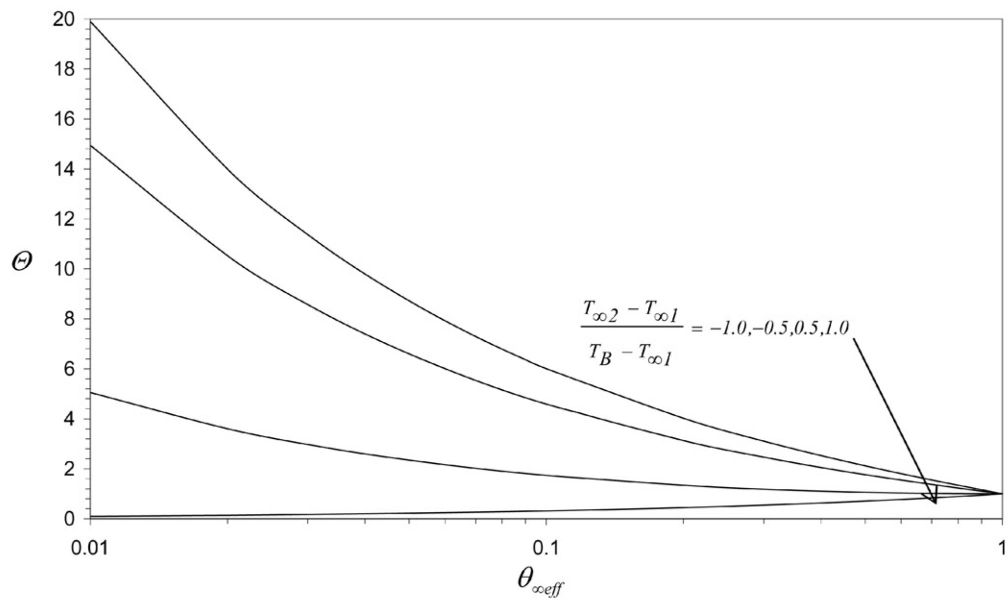

Fig. 11. Effect of effective free stream temperature $\theta_{\infty\ \mathrm{eff}}$ on the dimensionless fin heat transfer rate Θ for case E.

Table 1
Fin heat transfer rate for the analyzed Bi-convection fins

Case	Fin heat transfer rate q_f
A	$q_f = w\sqrt{(k_1t_1+k_2t_2)(h_1+h_2)}\{T_B - T_{\infty,\mathrm{eff}}\}$; $\frac{T_{\infty\ \mathrm{eff}}-T_{\infty1}}{T_{\infty2}-T_{\infty1}} = \frac{h_2/h_1}{1+h_2/h_1} + \frac{(Bi_2-Bi_1)/(1+h_2/h_1)}{[(\frac{h_1}{h_2})+1]+(\frac{k_1}{k_2})(\frac{t_2}{t_1})Bi_1+Bi_1}$; $Bi_1 = h_1t_1/k_1$; $Bi_2 = h_2t_2/k_2$
B	$q_f = N\{m_2H_2 - (m_2H_2+m_1H_1)\theta_{1B}\} + N\left\{\frac{(\frac{h_2}{h_1}-1)\coth(m_1H_1)-(\frac{h_1}{h_2}-1)\coth(m_2H_2)}{(\sqrt{\frac{h_2}{h_1}}\coth(m_1H_1)+\sqrt{\frac{h_1}{h_2}}\coth(m_2H_2))^2}\right\}$; $N = kt(T_{\infty1}-T_{\infty2})$; $\theta_{1B} = \frac{T_B-T_{\infty1}}{T_{\infty2}-T_{\infty1}}$; $m_1 = \sqrt{2h_1/(kt)}$; $m_2 = \sqrt{2h_2/(kt)}$
C	$q_f = M\left\{\frac{1+\sqrt{\frac{h_1}{h_2}}\tanh(m_1L_1)-(\frac{T_{\infty1}-T_{\infty2}}{T_{\infty1}-T_B})\,\mathrm{sech}(m_1L_1)}{\tanh(m_1L_1)+\sqrt{\frac{h_1}{h_2}}}\right\}$; $M = \sqrt{kA_Ch_1P_1}(T_B-T_{\infty1})$; $m_1 = \sqrt{h_1P/(kA_C)}$
D	$q_f = M_o\left\{\frac{h_2P_2}{h_1P_1} + \left(\frac{T_B-T_{\infty2o}}{T_B-T_{\infty1o}}\right)\right\}\Big/\sqrt{1+\frac{h_2P_2}{h_1P_1}} - \left\{\frac{aM_o/(T_B-T_{\infty1o})}{1+h_2P_2/[h_1P_1]}\right\}\sqrt{\frac{kA_C}{h_1P_1}} - \left\{\frac{bM_o/(T_B-T_{\infty1o})}{1+h_1P_1/[h_2P_2]}\right\}\sqrt{\frac{kA_C}{h_1P_1}}$; $M_o = \sqrt{kA_Ch_1P_1}(T_B-T_{\infty1o})$; $T_{\infty1} = T_{\infty1o}+ax$; $T_{\infty2} = T_{\infty2o}+bx$
E	$q_f = M\left\{\frac{h_2P_2}{h_{1\mathrm{eff}}P_1} + \left(\frac{T_B-T_{\infty2}}{T_B-T_{\infty1}}\right)\right\}\Big/\sqrt{1+\frac{h_2P_2}{h_{1\mathrm{eff}}P_1}}$; $h_{1\mathrm{eff}} = h_1+\rho c_P V$

4. Conclusions

Heat transfer through Bi-convection fins was modeled and analyzed analytically in this work. The "Bi-convection fin" is a terminology used in this work to refer to a fin subject to two different convective media. Five different cases were studied in this work: case A: Bi-convection thickness-wise Bi-metallic fins, case B: Bi-convection span-wise rectangular fins, case C: Bi-convection longitudinal-wise fins, case D: Bi-convection perimeter-wise fins with uniform cross-section, and case E: Bi-convection perimeter-wise permeable fins.

The following important conclusions were withdrawn from this work

- The fin heat transfer rate can be minimized at a certain value of effective free stream temperature for cases A, D and E.
- The fin heat transfer rate can be minimized at certain values of specific relative span height and convection coefficient ratio for case B.
- The fin heat transfer rate can be maximized at certain values of fin indices and convection coefficient ratio and for case C.

- The heat transfer through Bi-convection fin increases as its effective thermal conductivity, cross-sectional area, fin indices and the difference between the base and the effective free stream temperatures increase.
- The fin heat transfer rate for each case is listed in Table 1.

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